Birational-Integral Extensions (Commutative Algebra and Algebraic Geometry)

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Birational-integral extensions

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We shall study extensions of rings, especially, integral extensions of rings. Let R be an integral domain and let A be a ring which is birational and integral over R. For this purpose, the seminormalization ${}_A^+R$ of R in A, defined as follows, plays an important role.

Definition. (i) ${}_A^+R = \{\alpha \in A \mid \alpha \in R_{\mathfrak{p}} + J(A_{\mathfrak{p}})$ for any $\mathfrak{p} \in \operatorname{Spec} R\}$, where $J(A_{\mathfrak{p}})$ is the Jacobson radical of $A_{\mathfrak{p}}$.

(ii) If $R = {}_A^+R$, then we call R seminormal in A.

(iii) ${}_A^wR = \{\alpha \in A \mid \alpha^{p^n} \in R_{\mathfrak{p}} + J(A_{\mathfrak{p}})$ for some non-zero integer n, where p is the characteristic of the residue field $k(\mathfrak{p})$, $\mathfrak{p} \in \operatorname{Spec} R\}$ is called the weak normalization of R in A.

The seminormalization and the weak normalization are characterazied by the following proposition.

Proposition 1. ${}_A^+R$ (resp. ${}_A^wR$) is the greatest subring R' of A such that $R' \supseteqq R$, and

(i) for any $\mathfrak{p} \in \operatorname{Spec} A$, there is exactly one $\mathfrak{p}' \in \operatorname{Spec} R'$ lying over $\mathfrak{p}$, that is, the canonical map

$$\operatorname{Spec} R' \longrightarrow \operatorname{Spec} R$$

is injective (especially, since R' is integral over R, this map is an isomorphism).

(ii) the canonical homomorphism

$$\mathbf{k}(\mathfrak{z}) \longrightarrow k(\mathfrak{z}')$$

is an isomorphism (resp. gives a pure inseparable extension), where $\mathbf{k}(\mathfrak{z})$ is the residue field of R .

Corollary 2. We have

(i) $^+_A(^+_AR) = {}^+_AR$, that is, ^+_AR is seminormal in A.

(ii) Let $R \subsetneqq B \subsetneqq {}^+_AR$. Then $^+_BR = B$.

The proof of this corollary is easy.

Definition. Let $\mathfrak{z}_1, \ldots, \mathfrak{z}_t$ be prime ideals of R and let P_{ij}, $1 \leqq j \leqq e_i$, be all prime ideals of A lying over $\mathfrak{z}_i$. We call a subring B of A the gluing of A with respect to $\{\mathfrak{z}_1, \ldots, \mathfrak{z}_t\}$ if

$$B = \{\alpha \in A \mid \alpha(P_{i1}) = \ldots = \alpha(P_{ie_i}) \in \mathbf{k}(\mathfrak{z}_i) \text{ for all } i\},$$

where $\alpha(P_{ij})$ is the residue class of α with respect to P_{ij}.

Put $P_i = \{\alpha \in B \mid \alpha(P_{i1}) = \ldots = \alpha(P_{ie_i}) = 0\}$. Then $P_i = P_{ij} \cap B$ for all j, P_i is the only prime ideal of B lying over $\mathfrak{z}_i$, and we have $\mathbf{k}(P_i) = \mathbf{k}(\mathfrak{z}_i)$.

Proposition 3. If B is given by the gluing of A with respect to $\{\mathfrak{z}_1, \ldots, \mathfrak{z}_t\} \subset \operatorname{Spec} R$, then B is seminormal in A and $\operatorname{Ass}_B(A/B) \subsetneqq \{P_1, \ldots, P_t\}$, where $\operatorname{Ass}_B(A/B)$ is the set of all the associated prime divisors of the B - module A/B .

Conversely, we have:

Theorem 4. If B is seminormal in A, then B is given by the gluing of A with respect to $\mathrm{Ass}_B(A/B)$.

The proof of the above results is shown in [2] and [3].

Therefore, we know that if A/R is an integral extension, then ${}^+_A R$ is given the gluing of A with respect to $\mathrm{Ass}_{({}^+_A R)}(A/{}^+_A R)$. Hence we ask what is the extension ${}^+_A R/R$. For convenience, put $C = {}^+_A R$. Then ${}^+_C R = C$, that is, the seminormalization of R in C is equal to C, itself.

Let φ_A be a homomorphism of A to $(A \otimes_R A)_{red}$ over R, such that

$$\bar{\varphi}_A : A \longrightarrow A \otimes_R A \longrightarrow (A \otimes_R A)_{red}$$
$$\alpha \longmapsto \alpha\otimes 1 - 1\otimes\alpha \longmapsto \overline{\alpha\otimes 1 - 1\otimes\alpha}\ .$$

M. Manaresi proved in [1] the following:

Proposition 5. $\mathrm{Ker}\ \bar{\varphi}_A = {}^W_A R$, that is, the kernel of $\bar{\varphi}_A$ is equal to the weak normalization of R in A.

In our situation, since $C = {}^+_C R$, we have $C = {}^W_C R$. Hence we have $\bar{\varphi}_C(C) = (0)$, i.e., $\bar{\varphi}_C$ is the trivial map. Let I_C be the kernel of the canonical homomorphism

$$\psi_C : C \otimes_R C \longrightarrow C\ .$$

Then I_C is generated by $\{\alpha \otimes 1 - 1 \otimes \alpha \mid \alpha \in C\}$. Since $\bar{\varphi}_C(C) = (0)$, that is, $\alpha \otimes 1 - 1 \otimes \alpha$ is nilpotent and I_C is a two-sided ideal of $C \underset{R}{\otimes} C$, we have that I_C is nilpotent, say $I_C^{q+1} = (0)$. Therefore we see that the q-th differential module Ω_C^q is isomorphic to $I_C / I_C^{q+1} = I_C$. Hence there exists a canonical q - th derivation of C over R such that

$$\begin{aligned} \Delta_q : C &\longrightarrow I_C \cong \Omega_C^q \\ \alpha &\longmapsto \alpha \otimes 1 - 1 \otimes \alpha \end{aligned}$$

and $\Delta_q^{-1}(0)$ is a subring of C containing R.

More generally, we have:

Proposition 6. Let N be a $C \underset{R}{\otimes} C$ - submodule of Ω_C^q (for example, I_C^t, where t is an integer). Then $\Delta_q^{-1}(N)$ is an intermediate ring between R and C.

Proof. Let α, β be any elements of $\Delta_q^{-1}(N)$. Then $\Delta_q(\alpha + \beta) = \Delta_q(\alpha) + \Delta_q(\beta) \in N$ and

$$\Delta_q(\alpha\beta) = \alpha\beta \otimes 1 - 1 \otimes \alpha\beta = (\alpha \otimes 1)(\beta \otimes 1 - 1 \otimes \beta) + (1 \otimes \beta)(\alpha \otimes 1 - 1 \otimes \alpha)$$

$\in N$.

Therefore we have $\alpha + \beta$, $\alpha\beta \in \Delta_q^{-1}(N)$.

In the paper[4], Lipman introduced the following notion: For a ring A and a subring B of A, we call

$$\underset{A}{*}B = \left\{ \alpha \in A \mid \alpha \otimes 1 = 1 \otimes \alpha \quad \text{in} \quad A \underset{B}{\otimes} A \right\}$$

the strict closure of B in A. If $B = \underset{A}{*}B$ then we say that B is strictly closed in A.

Using these notations, we have:

Proposition 7. In the above notation, $\Delta_q^{-1}(N)$ is strictly closed in C.

Remark. If D is a high order derivation, then $B = \{\alpha \in C \mid \alpha D = D\alpha\}$ is a subring of C and strictly closed. Especially, if D is a derivation (of 1 - st order) of C over R such that $D: C \longrightarrow M$, where M is a C - module, then $D^{-1}(0) = \{\alpha \in C \mid D(\alpha) = 0\} = \{\alpha \in C \mid D\alpha = \alpha D\}$ is a subring and strictly closed in C.

Proposition 8. Let A be a ring containing a field k and let $\mathfrak{m}$ be a maximal ideal of A and assume that $A/\mathfrak{m} \cong k$. Let Q be a primary ideal belonging to $\mathfrak{m}$ and assume that $Q \geqq \mathfrak{m}^{q+1}$. Then

$$\begin{aligned} \Delta: A &\longrightarrow \mathfrak{m}/Q \quad , \\ \psi \quad & \\ \alpha &\longmapsto \overline{\alpha - \alpha(\mathfrak{m})} \end{aligned}$$

where $\alpha(\mathfrak{m})$ is the residue class of α and we regard $\alpha(\mathfrak{m})$ as an element of k, hence of A.

Then Δ is a q-th order derivation and we have

$$R = \{\alpha \in A \mid \alpha\Delta = \Delta\alpha\} = k + Q.$$

Remark. Let A be a noetherian domain containing a field k and $\mathfrak{p}$ be a prime ideal of A. Then the completion $\hat{A}$ of A has a coefficient field. Let K be the coefficient field and let q be a primary ideal belonging to $\mathfrak{p}$. Hence we have a canonical high order derivation

$$\Delta : A \longrightarrow A_{\mathfrak{p}} \longrightarrow \hat{A}_{\mathfrak{p}} \longrightarrow \hat{\mathfrak{p}}/\hat{q} .$$

On the other hand, the following results are well known (see [5]).

Proposition 9. (i) ${}^{*}_{A}B = A$ if and only if the extension A / B is epimorphic ($\overset{def}{\Longleftrightarrow}$ if $B \xrightarrow{i} A \underset{f_2}{\overset{f_1}{\rightrightarrows}} D$ and $f_1 \circ i = f_2 \circ i$, then $f_1 = f_2$).

(ii) A / B is epimorphic and integral if and only if $A = B$.

For convenience, let $C = C_0$ and $C_1 = \Delta_q^{-1}(0)$. Then we have:

Proposition 10. If $C_0 \supsetneqq R$, then $C_0 \supsetneqq C_1 \supseteqq R$.

Proof. If $C_0 = C_1$, then we have ${}^{*}_{C}R = C$, hence $C_0 = C = R$, because the extension C / R is epimorphic and integral.

Continuing this process, we have:

Theorem 11. There exist a sequence of invariant subrings with respect to some high order derivations and an integer d such that

$$C_0 \supsetneqq C_1 \supsetneqq \cdots \supsetneqq C_d = R \quad .$$

Remark. In the above C_{i+1} is an invariant subring of C_i with respect to the canonical high order derivation over R, $\Delta_i\colon C_i \longrightarrow \Omega_R(C_i)$.

Lemma. If $I_C^t = I_C^s$ for $t > s$, then we have $I_C^s = (0)$.

Proof. Indeed, for $\ell \gg 0$, we have $I_C^\ell = (0)$. Hence

$$I_C^s = I_C{}^t = I_C^{t-s+s} = I_C^{2t-s} = \cdots = (0).$$

Therefore there

$$I_C \supsetneqq I_C{}^2 \supsetneqq \cdots \supsetneqq I_C^q \supsetneqq I_C^{q+1} = (0) \quad .$$

For the canonical high order derivations

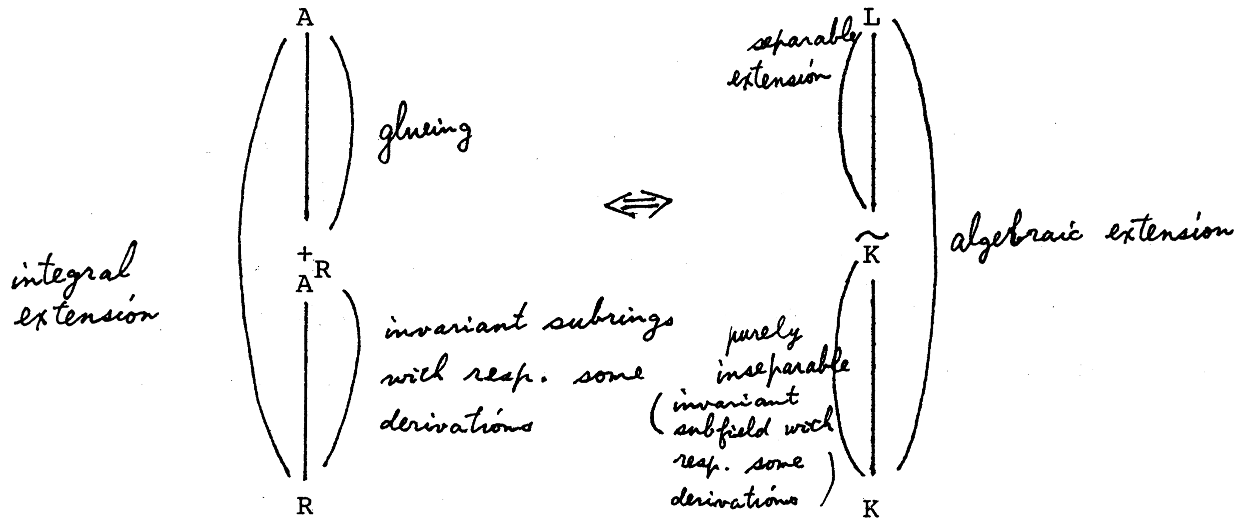
$$\Delta_i\colon C \longrightarrow I_C \,/\, I_C^i = \Omega_C^{i-1},$$

let $B_i = \Delta_i^{-1}(0)$. Then $B_i = \Delta_q^{-1}(I_C^i)$, and B_i is the invariant subring of B_{i-1} with respect to $\Delta_{i|B_{i-1}}$. Since $\Delta_{i|B_{i-1}}$ is a (1-st order) derivation, we have the following:

Theorem 12. There exists a sequence of invariant subrings with respect to some derivations and an integer g such that

$$C = C_0{}' \supsetneqq C_1{}' \supsetneqq \cdots \supsetneqq C_g{}' = R.$$

Therefore we say that the integral extensions are similar to the algebraic extensions of fields.



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